Improving Understanding in Integration with the Computer

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Often students' tendency in approaching calculus is to follow an algorithm or manipulate symbols. Mathematics educators seek to provide a range of experiences that develop mathematical ideas in a cognitive manner so that the learner both knows and understands. In this paper I describe how students used spreadsheets and symbolic manipulators to investigate the processes and concepts of integration, using the modules of work to supplement traditional approaches to integration. They gained a significant improvement in proceptual understanding, especially with regard to misconceptions exhibited in the pre-test.

Introduction

Many students are gaining instrumental rather than a relational understanding (Skemp, 1976) in calculus, learning 'how to' rather than 'why'. They have experienced difficulties with calculus by relying on formalised rules, procedures and calculus algorithms, disregarding conceptual understanding. When the students are confronted with a complex definition, many conceptual problems may arise such as with infinite, logical and manipulative processes. The conceptual difficulties with limit, differentiation and integration have been explored (see e.g. Tall, 1986; Steen, 1988; Barnes, 1988; Li & Tall, 1993; Thompson, 1994) and considerable effort has also been put into finding ways to improve such understanding, often using computer software such as symbolic manipulators (e.g. Small & Horsack, 1986; Palmiter, 1991; Barnes, 1994; Hubbard, 1995). In a previous paper, Thomas & Hong (1996) have represented the sort of misconceptions which many students have in calculus. The study described here investigated student thinking and misconceptions when dealing with integration; how the students using spreadsheets and symbolic manipulators had significantly enhanced conceptual understanding.

Background

Process and Concepts

Robert & Schwarzenberger (1991) described abstraction as involving the recognition of objects and properties which not only apply to the objects from which a generalisation is made but also to any other objects which obey to the same properties. Estimating the area under a curve with sums and passing to a limit is a process. Dubinsky & Lewin (1986) describe the *encapsulation* of such a process as an object. Students who seem to understand the process of integration often have difficulty with the next step of varying, the upper limit of the integral to obtain a function. They lack the ability to encapsulate the entire area process into an object which could then vary as one of its parameters varies. The definite integral forms an important example that can be interpreted as encapsulation together with interiorisation (Dubinsky, 1991). Gray & Tall (1993) have introduced symbolism and define the notion of procept as an amalgam of three things - process, symbol and concept. Procepts are first met through a process, then a symbolism is introduced for the product of that process. It seems that much of the symbolism used in mathematics carries for the mathematician the dual role of process and concept. This distinction between the usage of symbolisation to stand for a process or a concept or conceptual structure depending on one's point of focus is clearly an important one mathematically.

The Role of the Computer in Calculus

It is possible that the computer experiences (Ayers *et al*, 1988) may help students. If a student implements a process on a computer, using software that does not introduce programming distractions then the students tend to interiorise the process. If the same process can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process. So computer experiences encouraging reflective abstraction may be constructed in a wide variety of areas of mathematics. Tall (1993) suggests that the computer relieves the learner of the difficulty of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it may become possible for the learner to explore the properties of the object brought forth by the process, before, or at the same time, as studying the process itself. This new adaptability in curriculum progression he has called the *principle of selective construction.* Tall & Thomas (1991) described a *versatile* mathematician as having the ability to think in terms of processes or concepts. It is our contention that using computer software in mathematics courses can encourage the students' understanding of processes, thus facilitating versatility. Monaghan (1986) emphasised how a process creates a *generic limit* concept in which the varying process is encapsulated as an indefinite *variables* object such as $0.\dot{9}$. Monaghan (1993) also studied the growth of 16/17 year old students' conceptualisation of real number, limit and infinity over one year. CAS (Computer Algebra System) students using the symbolic manipulator software *Derive* were better able to see the limits as objects than traditional students. Another link with procepts is that some students used x^n and nx^{n-1} as generic terms for polynomials and their derivatives and questioned the functional notation led by their teacher. Computer software can evaluate many limits, so the possibility arises that it may allow the curriculum to give a more balanced view of limit as concept and process by early focus on the limit as concept with the computer carrying out the process internally.

Method

We devised a test which aimed to measure the students' conceptual understanding connected with integration, and which comprised two sections. The 10 questions of section I were focussed on the process-oriented algorithms which any student of integration could be expected to know. For example, they were asked to integrate, using antidifferentiation techniques $\int f(x)dx$ =F(x)+C. For the questions of section II, a deeper understanding of concepts was introduced. The 13 questions of section II were aimed at concepts rather than processes. Some questions in sections I & II were linked to see if students had developed techniques of algorithms but did not have the corresponding concepts, or the ability to apply the techniques when solving problems. The questions in section II cannot simply be calculated without conceptual understanding but those of section I could be. The categories of section I & II applied were as follows: conservation of integral, the maximum values of an integral function, the definite integral of a function that crosses the x-axis and area, summation using sigma (Σ) & Riemann sum, Riemann integral, sketching the integral function. The test was tried twice, the pre-test was attempted after formal learning of integration in calculus. After that the computer tutorial was provided, the post-test was used to re-evaluate any improvement. The questionnaire was given to 36 (age 16~19 years) high school students in 1996, comprising 14 female and 22 male students. The same test questions were given in the pre-test and the post-test for the same students.

Using modules based on the Excel spreadsheet and Maple symbolic manipulator, the processes of integration were investigated for 6~7 one hour sessions. The Excel spreadsheet was used for four one hour sessions and the Maple symbolic manipulator for 2~3 one hour sessions. The students already had a basic knowledge of the Excel spreadsheet but it was only in operational, not mathematical aspects. The aim of these computer tutorials was to see if an improvement in the understanding of the concepts associated with integration was produced by giving the students direct experience of experimentation with the processes which lead to them. 17 of the students were individually interviewed to investigate further their understanding. They consisted of students who had: greatly improved; made some improvement; or others who showed no improvement in their understanding. Comparing their results on both tests, they were questioned in their interview about their view of integration and their practical experiences. The recorded tapes of the interviews were transcribed and used for the analysis of the students' understanding.

Results

Misconceptions

Common misconceptions of students based on the results of the pre-test were categorised as; single value integral misconception, inversion misconception, general function misconception, and integration process disregarded misconception.

• Single Value Integral Misconception

In response to the question:

Section II. Find the interval [a, b] for which the value of the integral $\int_a^b (2-x-x^2)dx$ *is a maximum.*

the single value integral misconception appeared as follows. The concept of maximising an area function is more difficult than calculating a single area, however, it does give students opportunity to demonstrate their ability to extend their algorithmic knowledge and apply it in problem contexts, since it involves seeing a, b as variables. Some students wrote the following in answer to this question:

$$\left[2x-\frac{1}{2}x^2-\frac{1}{3}x^3\right]_a^b = 2b-\frac{1}{2}b^2-\frac{1}{3}b^3-(2a-\frac{1}{2}a^2-\frac{1}{3}a^3).$$

It seems that such students are locked into a process-oriented view and unable to proceed beyond the algorithm they have instrumentally learned and are unable to recognise the interval [a, b] as representing a variable area. Rather they are only able to compute a single value. Hence we call this the *single value integral misconception.* They don't see $\int$ as a function, but as Dubinsky described (see above), they lack the ability to encapsulate the area process into an object. Another case of this misconception was that a and b were sometimes replaced by values such as 1, 2; $\int_1^2 (2x-\frac{1}{3}x^3)dx$, again showing that the students depended on a restricted algorithm.

• Inversion Misconception

Here students responded to the integral: $\int_a^b f(x)dx = c$ by writing $\int_b^a f(x)dx = \frac{1}{c}$ or $-\frac{1}{c}$.

This misconception applies where the limits were reversed under the changed variable, we call it the *inversion misconception*. It was manifested in response to the question,

Section II. Given that $\int_9^{16} \sqrt{x}dx = \frac{74}{3}$, what is $\int_{16}^{9} \sqrt{t}dt$?

when the students inverted the answer given obtaining $\frac{3}{74}$, or gave $-\frac{3}{74}$. They seem to have applied an inappropriate algorithm obtained from *inversion of limits,* namely that they considered inversion of limits inverted the value.

• General function Misconception

Students were often unable to cope with the generalised function f(x) or f(t-1) since they could not apply a procedure. When considering the notation *f(t-1), f(x)+2,* some students misconceived the *t-1, x* as a variable to integrate or used *ft-f, ft-1, fx+2* as functions to integrate, with f as a constant. For example,

Section II.8. If $\int_a^5 f(x)dx = 63$*, and the graph of f(x) is shown below, what is a possible value of 'a'?*

Some students never used the given coordinates (5, 30) of the graph, instead they considered the function f(x) as x and obtained; $\left[\frac{x^2}{2}\right]_a^5 = 63$ i.e. $x \to \frac{x^2}{2}$

Another example was when dealing with:

Section II.9. If $\int_1^5 f(x)dx = 10$*, then write down the value of* $\int_1^5 (f(x)+2)dx$.

One student regarded the function *f(x)*+2 as *fx*+2 so trying to find a constant f, he substituted *fx*+2 and took the integral of it. Fortunately the answer was correct!

$$\left[\frac{f}{2}x^2\right]_1^5=10,\ f=\frac{5}{6},\ \left[\frac{5}{6}x^2\times2+2x\right]_1^5=18$$

Section II.6. *If* $\int_1^3 f(t)dt=8.6$, *then write down the value of* $\int_2^4 f(t-1)dt$.

The general function misconception appeared here as:

$$\left[\frac{f}{2}t^2-t\right]_1^3=8.6,\ f=2.15,\ \left[\frac{2.15}{6}t^2-2.15t\right]_2^4=8.6$$

Considering the notation f(t-1) the student misconceived the t-1 as a variable to integrate. i.e as the dependent variable rather than independent. Other examples were:

$$\left[\frac{1}{2}t^2-t\right]_2^4=4-0=4 \qquad \text{or} \qquad \int_2^4 f(t-1)dt=\int_1^3 f(t)dt\ -1=7.6$$

In every case above the students' known processes did not allow them to deal with functions such as f(t-1) and so they used not concepts, but incorrect procedures to deal with their lack.

- Integration Process Disregarded

$$\int_a^b f(x)dx \quad \Rightarrow \quad [f(x)]_a^b$$

Another way students coped with their inability to work with f(x) was to take the interval [a, b] and substitute it into the given function without integration; for example; $\int_1^4 (x-1)^2dx=\left[x^2-2x+1\right]_1^4$.

The above examples of misconceptions show that students conceived integral as an instruction to carry out a process or algorithm not as the encapsulation of the limit of area sum. Also integral calculus was considered as a series of processes with related algorithms, without adaptable ideas.

Statistical comparison between pre- & post-test

Overall: A statistical analysis of the student' answers to the pre & post test was conducted to see if there was any significant difference in the aspects of process-oriented skills (section I) and conceptual understanding or proceptual thinking (section II) displayed by the students. The results of the t-test show that there is a significant difference for the results of the 36 students in proportions of correct answers of the pre and post test;

Section I: $\overline{X}_{pre}=0.44$, $\overline{X}_{post}=0.57$, $n_{pre}=n_{post}=36$, t=3.94, p< 0.001

Section II: $\overline{X}_{pre}=0.15$, $\overline{X}_{post}=0.32$, $n_{pre}=n_{post}=36$, t=5.19, p< 0.001

Although there is an improvement in both the process ability and the conceptual understanding of the students, the computer tutorial seems to have considerably improved the conceptual understanding as represented in the questions of section II compared with before the computer tutorial.

Gender Consideration: The results of the 36 students, 22 males and 14 females who were analysed for any gender influence in the computer work. A one-tailed t-test was used to investigate whether there was a significant improvement in the students' results after having computer tutorials. The males performed significantly better on the post-test than they did before the computer work (Section I: $\overline{X}_{pre}=0.43$, $\overline{X}_{post}=0.55$, $n_{pre}=n_{post}=22$, t=2.72, p< 0.01; Section II: $\overline{X}_{pre}=0.17$, $\overline{X}_{post}=0.30$, $n_{pre}=n_{post}=22$, t=3.23, p< 0.005). Also, the performance of the females was significantly higher than on the pre-test (Section I: $\overline{X}_{pre}=0.46$, $\overline{X}_{post}=0.61$, $n_{pre}=n_{post}=14$, t=2.86, p< 0.01; Section II: $\overline{X}_{pre}=0.11$, $\overline{X}_{post}=0.35$, $n_{pre}=n_{post}=14$, t=4.39, p< 0.001). This indicates that for both genders students' proceptual understanding was enhanced after having computer tutorials. However there was no significant difference in the results between the genders.

Individual question analysis: The individual question results were analysed on the basis of matched questions of section I & II, such as conservation of integral, the maximum values of an integral, transformation parallel to x, y-axis, as table 1 shows. Since it was hoped that there would be improvement, a one-tailed z-test was applied.

Table 1: Proportions of students giving correct answers

Concept and Section		$\hat{p}_{pre}$ (n=36)	$\hat{p}_{post}$ (n=36)	z	p
Conservation of Integral	Section I.1	0.69	0.72	0.26	n.s
	Section II.4	0.22	0.47	2.31	< 0.05
The maximum values of an integral	Section I.2	0.61	0.75	1.28	n. s
	Section II.3	0.00	0.17	2.68	< 0.005
Transformation parallel to y-axis	Section I.4 b)	0.72	0.89	1.83	< 0.05
	Section II. 9	0.11	0.44	3.40	< 0.001
Transformation parallel to x-axis	Section I.5	0.47	0.69	1.96	< 0.05
	Section II.6	0.11	0.50	3.95	< 0.001

We can see that the students who had used the computer performed considerably better afterward in the aspect of the conceptual questions indicated above. Moreover, student knowledge of the concept of transformation improved greatly in both sections. It means that the students had successfully achieved the bilateral aspects of process-oriented skills and conceptual understanding. One example, question 10 from section II, involving Riemann sums, was as follows:

If f(x) is a strictly decreasing function, an approximation to $\int_1^4 f(x)dx$ using a Riemann sum (ie. of rectangles under the curve) with left end points and number of strips n=10 works out to be 7.615. Will an approximation with left end points and number of strips n=50 be:

(a) more than 7.615 (b) less than 7.615

(c) equal to 7.615 (d) less than or equal to 7.615

(e) not possible to say (f) more than or equal to 7.615

Explain your answer.

Analysing this revealed that the students improved in their understanding (section II.10: $\hat{p}_{pre}$=0.14, $\hat{p}_{post}$=0.50, n_{pre}= n_{post}=36, z=3.56, p<0.001). Many case of errors relating to Riemann integral involved the idea that left (upper) sum always increases as the number of strips increases regardless of whether it was a decreasing function. However the computer students obtained the idea that the value of the left (upper) sum decreases as the number of strips increases for a strictly decreasing function. Thus, regardless of gender, there were the significant improvements on students' understanding of process-oriented and conceptual integration for almost every question on the test.

Case study

Some of the students showed an excellent understanding in the post-test following the computer work. From now on I will discuss the work of one student who greatly improved in the aspect of conceptual understanding on section II. His proportion of correct answers in sections I & II were 8/10, 2/13 in pre-test, 10/10, 11/13 in post-test. For each of section I & II, the improved proportions of correct answer were as much as 20%, 71.6%. Improvements confirmed in the interviews after the tutorial included:

Interviewer: Has this tutorial affected your understanding of integration? If so how?
Student: Yes, I learnt more about finding integrals by summing rectangles.
Interviewer: Which section was useful to you? How?
Student: Riemann sum (leftsum, rightsum)

This shows that he thought he understood better than before. His solutions confirmed this. For example, one of the improved cases was the concept of maximum value where his work changed as follows:

The maximum value		Pre - Test	Post -Test
Section II	$\int_a^b (2-x-x^2)dx$	Single value integral misconception: He factorised, and substituted x=-2, 1, answered [1, $1\frac{1}{6}$]: $\int(2-x-x^2)dx = 2x-\frac{1}{2}x^2-\frac{1}{3}x^3$, $2(-2)-\frac{1}{2}(-2)^2-\frac{1}{3}(-2)^3=-3\frac{1}{3}$, $2(1)-\frac{1}{2}(1)^2-\frac{1}{3}(1)^3=1\frac{1}{6}$, so [1, $1\frac{1}{6}$]	He exactly solved by factorising: $2-x-x^2$=0, (x+2)(x-1)=0, x=-2 or x=1

His error in the pre-test involved the *single value integral misconception'*, the given function was factorised correctly, but the values x=-2, 1 were substituted for a, b to find the maximum value. He considered the maximum value as between x=1 and x=$1\frac{1}{6}$. However, he showed his revised understanding following the computer work as shown above, and solved correctly. For the question of transformations parallel to the x-axis he gave no attempt, in the pre-test. After the computer work, his improved concept that the translation of the graph leaves the area unchanged was shown as follows:

Transformation parallel to x-axis		Pre - Test	Post -Test
II.6	$\int_2^4 f(t-1)dt$	No attempt	He correctly made a relation as follows: $\int_2^4 f(t-1)dt = \int_1^3 f(y)dy$

In his interview, he demonstrated that he had an improved conception of transformation;

Interviewer: If we compare f(x)= x^2, with h(x)= $(x-3)^2$, what is the effect on the graph?
Student: It is moved right 3 units.
Interviewer: If we compare $\int_0^2 x^2dx$ with, $\int_3^5 (x-3)^2dx$, what's the change?
Student: Equal to $\int_0^2 x^2dx$

Looking at his concept of Riemann sum, while he simply made a guess or no attempt before the computer tutorial, after he was certain of the concept. In the pre-test he selected a) by guessing, admitting this in the interview. In the post test he correctly selected, and his answer to the Riemann integral was improved as follows:

Riemann Integral	Pre - Test	Post -Test
II.5 Rightsum -Leftsum = 0	No attempt	He exactly selected.
II.10 Riemann sum	No attempt	He correctly answered: 'less than 7.615'
II.12 $\sum_{i=0}^{9} f(x_{i+1})\cdot\frac{3}{10}$	He chosen the diagram of a) by guess.	He correctly selected the diagram b)

This improvement was confirmed in his interview, along with an understanding of limit:

Interviewer: For the number of equal sub intervals as $n \to \infty$, what's the value of the width? Will it be smaller, larger or 0? What about the limit of the width?
Student: 0, 0

... Interviewer: What's the meaning of the $\lim_{\Delta x\to 0}\sum_{i=1}^{n} f(x_{i+1})\,\Delta x$ and $\lim_{\Delta x\to 0}\sum_{i=1}^{n} f(x_i)\,\Delta x$? How do they differ?

Student: Rightsum, leftsum

His interview showed a good understanding of the concept of antiderivative function too:

Interviewer: For example, x^2, x^2+3 and x^2-2 are all indefinite integrals of $2x$. Does it matter which of these you use when calculating $\int_a^b 2x\,dx$?

Student: No, they're all correct, but x^2 may be more correct because it could save confusion.

Interviewer: What's the meaning of 'antiderivative (indefinite) integral' and 'definite integral'?

Student: Antiderivative integral give a function giving area under the graph for any x-value, definite integral give area under the graph for a particular x value

He had a noticeable improvement in the concept of definite integral, antiderivative function, Riemann sum, Riemann integral, and transformation about the x-axis. Comparing the results of the pre-test and post-test, the computer work led to a great improvement, especially, in the aspect of conceptual understanding, and his interview showed his clearer thinking.

Conclusions

In the pre-test, students' attempts at solving section II problems where there is no clear process to execute exhibited gaps in their conceptual understanding. The cases of misconceptions and errors described here show students' understanding restricted to algorithms and process, lacking adaptable thinking. For example, $\int f(x)dx$ was seen as an instruction to carry out a process or algorithm not as the encapsulation of the limit of area sums. To improve the cognitive base for a flexible proceptual understanding of limits, areas and Riemann integral, suitable uses of computer software may be applied. The results of the post-test showed that the students' lack of understanding of concept was considerably improved. Evidence was provided by individual analysis, interview, and statistical analysis. The excel spreadsheet and symbolic manipulator allowed students to see what is included at each stage of the process, providing opportunities understanding of the concepts as well as the ability to calculate integrals. Specifically, the graphical aspects of concepts such as definite integral, transformations parallel to x, y-axis could be developed in a classroom, avoiding a passive style. The opportunity to study integration using the computer had given valuable perception into the processes of integration and this is in turn assisted the student's conceptual understanding. The experiment showed that when using the computer there was a significant development in conceptual understanding of concepts traditionally regarded to be of higher grade, and evidence of a more versatile form of thinking related to the computer experiences.

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